Matrices and Determinants

174. $\Delta(x)=\begin{vmatrix}\alpha+x & \theta+x & \lambda+x\\ \beta+x & \phi+x & \mu+x\\ \gamma+x & \psi+x & v+x\end{vmatrix}$

$$\Delta'(x)=\begin{vmatrix}1 & \theta+x & \lambda+x\\ 1 & \phi+x & \mu+x\\ 1 & \psi+x & \nu+x\end{vmatrix}+\begin{vmatrix}\alpha+x & 1 & \lambda+x\\ \beta+x & 1 & \mu+x\\ \gamma+x & 1 & \nu+x\end{vmatrix}+\begin{vmatrix}\alpha+x & \theta+x & 1\\ \beta+x & \phi+x & 1\\ \nu+x & \psi+x & 1\end{vmatrix}$$

Applying $C_2\to C_2-xC_1$ and $C_3\to C_3-xC_1$ in first and $C_1\to C_1-xC_2, C_3\to C_3-xC_2$ in second and $C_1\to C_1-xC_3$ and $C_2\to C_2-xC_3$ in third to get $\Delta'(x)=\begin{vmatrix}1 & \theta & \lambda\\ 1 & \phi & \mu\\ 1 & \psi & \nu\end{vmatrix}+\begin{vmatrix}\alpha & 1 & \lambda\\ \beta & 1 & \mu\\ \gamma & 1 & \nu\end{vmatrix}+\begin{vmatrix}\alpha & \theta & 1\\ \beta & \phi & 1\\ \gamma & \psi & 1\end{vmatrix}$

$\Rightarrow \quad \Delta''(x)=0$

If S is the sum of all the cofactors of all elements in $\Delta(0)$, then it can be seen that $\Delta'(x)=S$

On integrating $\Delta(x)=Sx+c$, we get $\Delta(0)=0+c$

Hence, $\Delta(x)=Sx+\Delta(0)$.

175. We have $|A-\lambda I|=\begin{bmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{bmatrix}=(\cos\theta-\lambda)^2+\sin^2\theta$

Therefore, characteristic equation of A is $(\cos\theta-\lambda)^2+\sin^2\theta=0$ or $\cos\theta-\lambda=\pm i\sin\theta$

Or $\quad \lambda=\cos\theta\pm i\sin\theta$ which are of unit modulus.

176. Let $A=[a_{ij}]$ be a given square matrix. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the characteristic roots of A. If $\phi(\lambda)$ is the characteristic function, then $\phi(\lambda)=|A-\lambda I|=\begin{vmatrix}a_{11}-\lambda & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22}-\lambda & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda\end{vmatrix}$

$=(-1)^n[\lambda^n+p_1\lambda^{n-1}+p_2\lambda^{n-2}+\ldots+p_n] \qquad \ldots$ (i)

$=(-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n) \qquad \ldots$ (ii)

On putting $\lambda=0$, we have $\phi(0)=|A|=\lambda_1\times\lambda_2\times\lambda_3\times\ldots\times\lambda_n=(-\lambda)^n P_n \qquad \ldots$ (iii)

Hence, the product of characteristic roots of a square matrix is equal to the determinant of the matrix.

177. Let A be an eigen value of A and X be a corresponding eigen vector. Then $AX=\lambda X$

Or $\quad X=A^{-1}(\lambda X)=\lambda(A^{-1}X)$ or $\frac{1}{\lambda}X=A^{-1}X \qquad [\because \quad$ A is nonsingular $\Rightarrow \lambda\neq 0]$

Or $\quad A^{-1}X=\frac{1}{\lambda}X$

Therefore, $\frac{1}{\lambda}$ is an eigen value of A^{-1} and X is the corresponding eigen vector.

178. Since α is a characteristic root of a nonsingular matrix, therefore $\alpha \neq 0$. Also α is a characteristic root of A implies that there exists a non-zero vector X such that $AX = \alpha X$

Or $(adj\ A)(AX) = (adj\ A)(\alpha X)$ Or $[(adj\ A)A]\ X = \alpha(adj\ A)X$

Or $|A|\ IX = \alpha(adj\ A)X$ $[\because\ (adj\ A)\ A = |A|\ I]$

Or $|A|\ X = \alpha(adj\ A)X$ or $\dfrac{|A|}{\alpha} X = (adj\ A)\ X$ Or $(adj\ A)\ X = \dfrac{|A|}{\alpha} X$

Since X is a nonzero vector, $\left|\dfrac{A}{\alpha}\right|$ is a characteristic root of the matrix adj A.

179. Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $\Rightarrow$ $A - \lambda I = \begin{bmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{bmatrix}$

$\Rightarrow$ $\det(A - \lambda I) = (a_1 - \lambda)[(b_2 - \lambda)(c_3 - \lambda) - b_3 c_2] - b_1[a_2(c_3 - \lambda) - c_2 a_3] + c_1[a_2 b_3 - a_3(b_2 - \lambda)]$

Now if one of the eigen values is zero, one root of λ should be zero.

Therefore, constant term in the above polynomial is zero.

$\Rightarrow$ $a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 c_2 a_3 + a_1 a_2 a_3 - c_1 a_3 b_2 = 0$

But above is the value of determinant of A. Hence, det A = 0.